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On the nonlinear extension of quantum superposition and uncertainty principles

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Abstract

We show that nonlinear extensions of quantum mechanics exist in which (extensions of) quantum superposition and uncertainty principles hold.

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1. Introduction

It was shown in [12,13] that a strong form of the Heisenberg uncertainty principle and a carefully selected set of reasonable mathematical hypothesis on the algebraic structure of the set of observables (associativity and algebraic closure) are sufficient to force the choice of *Projective Quantum Mechanics*. This expression will denote in this work quantum mechanics (QM) seen as a geometry of the projective Hilbert space. This theory, a synthetical description of which may be found in [12], has the same physical content of the ordinary Hilbertian formulation of QM without superselection rules, but it avoids the annoying and conceptually misleading "up to phase" language.

In spite of the fundamental role played by the observable algebra in the C^*-algebraic formulation of QM, we give up in this work the algebraic structure of the observable set

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and only take as fundamental the quantum superposition principle (QSP) and the Heisenberg uncertainty principle (HUP). By abstracting from the usual geometric formulation of quantum mechanics we select a minimal geometric formalism allowing
- the formulation of QSP,
- a complete characterization of observables as those respecting superpositions,
- a complete characterization of dynamical vector fields as those whose flows respect superpositions (isometricity), and
- the formulation of HUP.

As we will see a considerable part of the quantum formalism, which is necessary to reproduce the algebraic structure of QM, plays no role in the description of quantum fundamental principles. We only need a connection (to describe superpositions), a Finslerian structure (to define dispersions) and a Poissonian structure (to formulate HUP).

This work is the first step towards a geometrical foundation of nonlinear quantum mechanics (NLQM). Here we just identify some geometric structures which have to be (and can be) saved in attempts to delinearize ordinary quantum mechanics. However, superposition and uncertainty principles and the probabilistic content of QM are strictly linked to spectral theory of observables. So we have to study also a suitable spectral theory for nonlinear observables. This will be the argument of a forthcoming publication. Of course, at this stage of development of the delinearization program we do not claim a complete geometrical framework for nonlinear extensions of quantum mechanics.

1.1. Why nonlinear quantum mechanics?

Constraints make the (pure) state space of classical systems (phase spaces) strikingly nonlinear. But Heisenberg uncertainty inequalities forbid true quantum contraints, so the quantum state space appears (up to a phase) absolutely linear. Moreover, while observables and dynamical vector fields have no fundamental (logical) restriction in classical physics, they are subjected in QM to the constraint of respecting superpositions. In spite of the enormous foundational work justifying this contraposition, all the history of quantum physics is crossed by proposals aiming to delinearize at least some aspects of the quantum formalism. As usual there is a variegated gallery of motivations. De Broglie (see [15] and references therein) refusal of indeterminism is (as far as we know) the oldest and Weinberg [39] pragmatic request of experimentally testing linearity is perhaps the best known. Up to now no proposal aiming to delinearize QM at a fundamental level has been experimentally confirmed. Foundational objections have been raised against any possible nonlinear extension of quantum mechanics. Indeed Gisin [19], Peres [31] and others have remarked that undesirable consequences stem from introducing nonlinear dynamics in ordinary QM. Gisin shows how an EPR-like experiment combined with a simple nonlinear dynamics may be used to send faster-than-light signals. But it seems to us that he only shows how dangerous is acritically adopt the usual notion of mixed state in a nonlinear context. The necessity of consistently define such a notion accordingly to the given observable set has already been stressed in [27]. Analogously, as Weinberg has pointed out in [40], Peres proof that nonlinear dynamics necessarily violates the second law of thermodynamics only shows that von Neumann entropy has to be accurately reformulated in a nonlinear theory. So these objections are interesting but by no way decisive. On the other hand, there are at least two sectors where nonlinearities seem to play a relevant role: QM in a gravitational background and nonlinear approximations to linear quantum phenomena.

1.1.1. QM in a gravitational background

A crucial trend of modern theoretical physics is towards a quantization of gravitational fields, but serious objections have been formulated against this project. So it seems premature to restrict all the reaserch on this program. As Mielnik has pointed out all that physics seems to tell us is that *"[...] either the gravitation is not classical or quantum mechanics is not orthodox"* [27]. The heretical choice to look for a nonorthodox QM has been advocated in [24]. A theory in which gravitation is not classical *and* QM is not orthodox has on the contrary been advocated by Popova [32].

1.1.2. Nonlinear approximations to linear quantum phenomena

In classical mechanics a constraint is a surface in phase space, so it seems natural to think of a quantum constraint (QC) as a submanifold of the pure state space $\mathfrak{P}\mathcal{H}$. But since nonorthogonal quantum states are only partially distinguishable there is no physical way to implement a QC. The restriction of a dynamical vector field v to a submanifold S of $\mathfrak{P}\mathcal{H}$ gives us a vector field v_S which we may think as a nonlinear approximation of v. As a prototipical example of this situation we take Hartree–Fock equations. The restriction of dynamics from (the projective of) the antisymmetric Fock space to the (projective of the) 'space' of decomposable vectors [7] involves the loss of linearity but leaves us with an equation (the Hartree–Fock one) much more simple than the original Schrödinger equation. As Rowe [35] has shown Hartree–Fock dynamics may be described by a Hamiltonian equation on a Grassmanian manifold seen as a hypersurface of a projective Hilbert space. So Grassmanian manifolds and more generally coadjoint orbits of the unitary group (of whose the Grassmanian is an exceptionally simple example) are the first candidates to study nonlinearities.

1.1.3

It seems clever, before undertaking the study of truly nonlinear quantum systems, to understand better quantum dynamics on a coadjoint orbit and still before to understand better the geometry of ordinary (projective) QM. So in this paper our main aim will be to describe the interplay between the fundamental quantum principles and geometric structures used to describe quantum phenomena in ordinary QM. In Sections 2–5 we review these geometrical structures from this point of view. In Sections 6 and 7 we show how superposition principle and uncertainty principle can be extended to much more general contexts.

1.2. Fundamental quantum principles and the geometry of quantum mechanics

1.2.1. The superposition principle

One often claims that delinearization of QM necessarily entails the lost of the QSP [35]. Sometimes one even reads that the QSP cannot be formulated on the pure states space $\mathfrak{P}\mathcal{H}$, that is on the projective of the Hilbert space $\mathcal{H}$ [14]. All that stems from the widespread belief that the true nature of superpositions lies in the linear combination of wave functions. But as we will show, extending Cantoni's work [9], a neat geometrical formulation of the QSP is possible in $\mathfrak{P}\mathcal{H}$ by using the geodesic flow of the canonical (Riemannian) connection (see [12]). To be exact one must distinguish between strong superpositions of two states, corresponding to geodesics connecting these two states, and (ordinary) superpositions, corresponding to points of the smallest geodesically closed subset containing these two states. So the main ingredient for a neat formulation of the QSP is a geodesic flow, that we will introduce by means of a connection.

1.2.2. *Observables and dynamical vector fields*

Among geodesics a main role is played by the minimal (that is parametrized with arc length) ones. Indeed we will show that observables are nothing more than functions respecting minimal geodesics in the technically detailed meaning coded in the definition of geolinearity (see Definition 4.1). Moreover, we will characterize dynamical vector fields as those vector fields whose flows preserve geodesics and minimality (that is distance along geodesics). This is a second step in the comprehension of the geometry of quantum mechanics: there is a distinguished way to make superpositions, the one described by minimal geodesics.

1.2.3. *Dispersions*

The Riemannian structure plays a double role in the description of quantum mechanics. In the first place the Riemannian tensor determines a connection whose geodesic flow describes superpositions. In the second place the Riemannian structure may be seen as a Finslerian one (see [30] and for a recent overview [10]). By means of this Finslerian structure we introduce the notion of dispersion and characterize dynamical vector fields as those vector fields whose flow is isometric.

1.2.4. *Uncertainty principle*

Up to now no use has been made neither of the complex nor of the symplectic canonical structures of $\mathfrak{P}\mathcal{H}$ (see [12]). By introducing a weakly nondegenerate symplectic structure, but a Poissonian one would be sufficient, we may also describe the Heisenberg uncertainty principle in purely geometrical terms. The traditional formulation of the HUP (Heisenberg inequalities) amounts to requiring the continuity, in any point x, of the symplectic tensor with respect to the Riemaniann topology on the tangent space in x. Besides this kinematical formulation we develop a dynamical one totally equivalent to the former: the product of the (projective) velocities of two dynamical curves can never be lesser than the (cosine of the) intersection angle of the two curves. At last we characterize strong superpositions (geodesics) with respect to ordinary superpositions (geodesic envelope) as those points where Anandan–Aharonov time-energy uncertainty is minimal.

Concluding we stress that we are not proposing our abstract geometrical setting as a way to effectively extend quantum mechanics. A serious proposal cannot be based either exclusively or mainly on mathematical considerations. The starting point has surely to be a careful study of a well-selected phenomenology. We only want to point out that a generalized formulation is possible. Future research will tell us which modifications or restrictions have to be introduced. Our main aim will be to test the formalism on nonlinear approximations to quantum systems and to examine its usefulness in the study of mesoscopic systems.

1.3. Relations with previous works.

The idea to look at superpositions as projective geodesics has been stressed in the beautiful work [9]. To synthesize the mathematical differences between our study and Cantoni's one, we could say that while Cantoni based his work on the metric aspects of quantum formalism, our study is mainly of a differential-geometric nature. This difference is not purely formal. The richer differential-geometric formalism allows us a more complete geometric description of quantum formalism, above all, a complete characterization of observables and dynamic vector fields. This is essential in a study aiming to single out a class of manifolds suitable as candidate state spaces for nonlinear quantum systems. Moreover, it is relevant from a conceptual point of view to characterize in a different way observables and dynamical vector fields. These two concepts coincide in ordinary QM by virtue of Stone's theorem, but this is no longer true in open quantum systems and in many NLQM proposals. Furthermore, the characterization of observables as geolinear functions is a true implementation of linearity, while the characterization of dynamical vector fields as the Killing ones is an implementation of the notion of isometricity. The agreement between geolinearity and isometricity is a peculiar characteristic of projective QM which could be no more true in a nonlinear theory.

2. The geometric structure of quantum pure states space

In this section we describe, to fix terminology and notations, the elementary properties of $\mathfrak{P}\mathcal{H}$, that is the set of one-dimensional subspaces of a Hilbert space $\mathcal{H}$. We refer for details to [12,17].

Let $\mathcal{H}$ be a complex Hilbert space with inner product $(\ |\ )$ (it is trivial to verify that a great part of the results we shall state still hold when $\mathcal{H}$ is a real Hilbert space). We point out that we do not require neither finite dimensionality nor separability. We shall refer to $\mathfrak{P}\mathcal{H}$ as the projective of $\mathcal{H}$ and we shall use the symbol $\hat{\varphi}$ to denote the one-dimensional subspace generated by the nonzero element φ of $\mathcal{H}$. Moreover, we define: $\mathfrak{n}(\varphi) := \varphi/\|\varphi\|$ and $\mathfrak{p}(\varphi) := \hat{\varphi}$.

Definition 2.1. Let ψ be any normalized element of $\mathcal{H}$.
- $U_\psi := \{\hat{\varphi} \in \mathfrak{P}\mathcal{H};\ (\psi|\varphi) \neq 0\}$,
- $\mathcal{H}_\psi$ is the Hilbert space $\{\varphi\}^\perp$,

– $b_\psi : \hat{\varphi} \in U_\psi \rightsquigarrow Q_\psi \varphi/(\psi|\varphi) \in \mathcal{H}_\psi$ where $Q_\psi : \mathcal{H} \to \mathcal{H}_\psi$ is the canonical orthogonal projection,
– $\xi_\psi := (U_\psi, b_\psi, \mathcal{H}_\psi)$.

One proves without difficulty that ξ_ψ is a chart on $\mathfrak{P}\mathcal{H}$ and that the family of charts $\{\xi_\psi; \psi \in \mathcal{H}; \|\psi\| = 1\}$ is a holomorphic atlas on $\mathfrak{P}\mathcal{H}$. Moreover,

$$\mathfrak{p} : \mathcal{H} \setminus \{0\} \to \mathfrak{P}\mathcal{H} \tag{2.1}$$

is a holomorphic submersion and $\ker T_\varphi \mathfrak{p} = \mathbb{C}.\varphi$ for any normalized vector φ. This stems immediately from the following derivative:

$$D(b_\varphi \circ \mathfrak{p})(\varphi) = Q_\varphi. \tag{2.2}$$

Later on we shall make use of an arbitrarily fixed positive real number κ. We shall see later that, to have a correct correpondence with ordinary quantum mechanics, one must have $\kappa = \hbar$. Moreover, we shall use systematically the *sans serif* symbol to mean the local representative. Example: if v is a vector tangent in $\hat{\varphi}$ to $\mathfrak{P}\mathcal{H}$ then v is the local representative of v with respect to the chart ξ_φ. We point out that v depends on the choice of the normalized representative $\varphi \in \hat{\varphi}$.

Definition 2.2. For any $\hat{\varphi} \in \mathfrak{P}\mathcal{H}$ and $v, w \in T_{\hat{\varphi}}\mathfrak{P}\mathcal{H}$

$$g_{\hat{\varphi}}(v, w) := 2\kappa\Re(\mathsf{v}|\mathsf{w}), \tag{2.3}$$

where $\mathsf{v} := T_{\hat{\varphi}} b_\varphi(v)$, $\mathsf{w} := T_{\hat{\varphi}} b_\varphi(w)$ and $\varphi \in \hat{\varphi}$ is any normalized vector.

It is trivial to prove that (2.3) is well-formulated, i.e. that it does not depend on the choice of the representative φ of $\hat{\varphi}$. Moreover, g is a strongly nondegenerate (see [25]) Riemannian tensor: it is not difficult to verify that g is smooth and that, by virtue of the Riesz theorem, the map

$$g^\flat_{\varphi} : v \in T_{\hat{\varphi}}\mathfrak{P}\mathcal{H} \rightsquigarrow g_{\hat{\varphi}}(v, \_) \in T^*_{\hat{\varphi}}\mathfrak{P}\mathcal{H} \tag{2.4}$$

is a linear continuous isomorphism. As a holomorphic manifold $\mathfrak{P}\mathcal{H}$ is naturally endowed with a complex structure J. This structure is strictly linked with the Riemannian one:

Proposition 2.1. *$(\mathfrak{P}\mathcal{H}, g, J)$ is a strongly nondegenerate Kählerian manifold.*

We refer the reader for a proof of this well-known fact to [26, Proposition 5.3.1].

We close the section by collecting some relevant geometrical properties of $\mathfrak{P}\mathcal{H}$.

One proves easily that the map $\mathfrak{p} : \mathcal{H} \setminus \{0\} \to \mathfrak{P}\mathcal{H}$ is a Kählerian submersion. That is: for any nonzero $\varphi \in \mathcal{H}$ the restriction of the map $T_\varphi \mathfrak{p}$ to the orthogonal of its kernel is a $\mathbb{C}$-linear isometry.

As a Kählerian manifold $\mathfrak{P}\mathcal{H}$ has a symplectic structure. It is easy to see that the symplectic tensor satisfies the following relation:

$$\omega_{\hat{\varphi}}(v_{\hat{\varphi}}, w_{\hat{\varphi}}) = 2\kappa\Im(\mathsf{v}|\mathsf{w}). \tag{2.5}$$

3. The geodesic structure

3.1. The Riemannian connection

The (strongly nondegenerate) Riemannian tensor g of $\mathfrak{P}\mathcal{H}$ generates a vector-bundle isomorphism $g^\flat$ (with inverse $g^\sharp$) by means of which we may transport the canonical symplectic structure of $T^*\mathfrak{P}\mathcal{H}$ on $T\mathfrak{P}\mathcal{H}$. Let us introduce the function

$$K : v_{\hat{\varphi}} \in T\mathfrak{P}\mathcal{H} \rightsquigarrow g_{\hat{\varphi}}(v_{\hat{\varphi}}, v_{\hat{\varphi}}) \in \mathbb{R}. \tag{3.1}$$

The opposite $-v_K$ of the Hamiltonian vector field generated by K is called the canonical spray of $\mathfrak{P}\mathcal{H}$. For any chart ξ_φ there is one and only one smooth map

$$\Gamma : U_\varphi \to L^2_{\mathrm{sym}}(\mathcal{H}_\varphi, \mathcal{H}_\varphi) \tag{3.2}$$

such that

$$-\mathsf{v}_K(\eta, \mathsf{v}) = (\eta, \Gamma_\eta(\mathsf{v}, \mathsf{v})), \tag{3.3}$$

where v_K is the local representative of the vector field v_K with respect to the chart ξ_φ. The map Γ is called canonical bilinear symmetrical map determined by the spray $-v_K$. There is a general relation linking a Riemannian tensor g with the canonical map Γ associated with it [25]:

$$-\Gamma_\eta(\mathsf{v}, \mathsf{w}) = \tfrac{1}{2}\mathsf{g}^\sharp_\eta(D\mathsf{g}(\eta)(\mathsf{v})(\_, \mathsf{w}) + D\mathsf{g}(\eta)(\mathsf{w})(\mathsf{v}, \_) - D\mathsf{g}(\eta)(\_)(\mathsf{v}, \mathsf{w})).$$

By means of this relation one proves that:

Proposition 3.1. *The bilinear symmetric map associated to the tensor g on $\mathfrak{P}\mathcal{H}$ is (in the chart ξ_φ)*

$$\Gamma_\eta(\mathsf{v}, \mathsf{w}) := \frac{(\eta|\mathsf{v})\mathsf{w} + (\eta|\mathsf{w})\mathsf{v}}{1 + \|\eta\|^2}. \tag{3.4}$$

A proof may be found in [17].

3.2. The geodesic flow

We shall say that the vector $v \in T_{\hat{\varphi}}\mathfrak{P}\mathcal{H}$ is *2κ-normalized* if $g_{\hat{\varphi}}(v, v) = 1$. We point out that if v is 2κ-normalized then its local representative v is normalized.

Proposition 3.2. *Let $v \in T_{\hat{\varphi}}\mathfrak{P}\mathcal{H}$ be any 2κ-normalized vector. The geodesic tangent in $\hat{\varphi}$ to v is*

$$c_{\hat{\varphi},v}(t) = \mathfrak{p}(\varphi \cos t + \mathsf{v} \sin t). \tag{3.5}$$

The proof is a trivial application of the geodesic equation.

If $\hat{\varphi}$, $\hat{\chi}$ are distinct points of $\mathfrak{P}\mathcal{H}$ we shall use the symbol $\widehat{\varphi\chi}$ to denote the projective of the two-dimensional subspace of $\mathcal{H}$ generated by φ, χ (nonzero representatives of $\hat{\varphi}$ and $\hat{\chi}$, respectively). We shall say that $\hat{\varphi}$ and $\hat{\chi}$ are *antipodal* if $\varphi \perp \chi$ for any $\varphi \in \hat{\varphi}$, $\chi \in \hat{\chi}$.

Remark 3.1. A point $\hat{\chi}$ of $\mathfrak{P}\mathcal{H}$ belongs to $c_{\hat{\varphi}v}$ if and only if $\hat{\chi} \in \widehat{\varphi \mathsf{v}}$ and for some normalized vector $\varphi \in \hat{\varphi}$ (and then for all) one has

$$\frac{(\mathsf{v}|\chi)}{(\varphi|\chi)} \in \mathbb{R}. \tag{3.6}$$

Remark 3.2. For any pair of distinct non-antipodal points $\hat{\varphi}$, $\hat{\chi}$ in $\mathfrak{P}\mathcal{H}$ there is one and only one (up to reparametrization) geodesic crossing these two points. The equation of this geodesic is (up to reparametrization)

$$c_{\hat{\varphi}\hat{\chi}}(t) := \mathfrak{p}(\varphi \cos t + \mathfrak{n}[b_\varphi(\hat{\chi})] \sin t), \tag{3.7}$$

where $\varphi \in \hat{\varphi}$ is any normalized vector.

3.3. The exponential map and the injectivity radius

For any $\hat{\varphi} \in \mathfrak{P}\mathcal{H}$ and any (not necessarily 2κ-normalized) vector v in $T_{\hat{\varphi}}\mathfrak{P}\mathcal{H}$ let $c_{\hat{\varphi}v}$ be the only geodesic starting in $\hat{\varphi}$ with velocity v. We know from the Section 3.2 that

$$c_{\hat{\varphi}v}(t) = \mathfrak{p}(\varphi \cos(\|\mathsf{v}\|t) + \mathfrak{n}(\mathsf{v}) \sin(\|\mathsf{v}\|t)). \tag{3.8}$$

Hence the exponential map has the following form

$$\mathrm{Exp}_{\hat{\varphi}} v = \mathfrak{p}(\varphi \cos \|\mathsf{v}\| + \mathfrak{n}(\mathsf{v}) \sin \|\mathsf{v}\|). \tag{3.9}$$

Definition 3.1. Let (M, g) be any (strongly nondegenerate) Riemannian manifold. If x is any point of M, the *injectivity radius* of x (in symbols ι_x) is the supremum of all positive real numbers ρ such that $\mathrm{Exp}_x B_\rho(0_x)$ is injective, where $B_\rho(0_x)$ is the closed disc of radius ρ and center 0_x.

With standard arguments one shows:

Proposition 3.3. *The injectivity radius of any element $\hat{\varphi}$ of $\mathfrak{P}\mathcal{H}$ is $\iota_{\hat{\varphi}} = \kappa\pi$.*

Hence we have a neat interpretation of κ as (up to a trivial normalization) injectivity radius of any point. This interpretation has to be seen in contraposition with the usual one looking at κ as a measure of the holomorphic sectional curvature. Indeed one shows [22] that the holomorphic sectional curvature of $\mathfrak{P}\mathcal{H}$ is constant and that its value is $2/\kappa$. We think of this interpretation as being mathematically correct but physically misleading. Especially because the classical limit $\kappa \rightsquigarrow 0$ as no clear geometrical meaning. We point out that this limit is classical because, as we shall recall at the end of Section 5, to have an exact correspondence between projective QM and ordinary QM one is forced to take $\kappa = \hbar$. We believe that κ has to be thought as injectivity radius of pure states. Since injectivity radius measures total (physical) distinguishability (see the end of this section) *we may think of the limit* $\kappa \rightsquigarrow 0$ *as being an (infinite) extension of total distinguishability.*

3.4. The metric structure

Knowing the exponential map it is not difficult to derive the form of the metric structure:

Proposition 3.4. *The metric structure of* $\mathfrak{P}\mathcal{H}$ *has the following form: for any* $\hat{\varphi}, \hat{\chi} \in \mathfrak{P}\mathcal{H}$

$$d(\hat{\varphi}, \hat{\chi}) = \sqrt{2\kappa} \arccos |(\varphi|\chi)|, \tag{3.10}$$

where $\varphi \in \hat{\varphi}$, $\chi \in \hat{\chi}$ *is any pair of normalized representatives.*

We point out that the proof of this proposition is easy but not immediate. Indeed since Hopf–Rinow theorem does not hold on infinite-dimensional manifolds one has to prove by direct inspection that for any pair of points there is a minimal geodesic whose length is minimal among curves connecting these two points.

Exploiting the geodesic and the metric structures we see that $\mathfrak{P}\mathcal{H}$ belongs to a relevant class of manifolds:

Definition 3.2. A strongly nondegenerate Riemannian manifold (M, g) is an SC_l*-manifold* (for some positive real number l) if
- any geodesic is a periodic orbit of period l,
- any geodesic is *simple* (injective) when restricted to $[0, l[$.

We recall that SC_l is a particular class of manifolds all of whose geodesics are closed. We refer the reader to [6] for a description of the relations linking this class to other classes of manifolds with closed geodesics.

Indeed one has the following result.

Proposition 3.5. $\mathfrak{P}\mathcal{H}$ *is an* $SC_{\pi\sqrt{2\kappa}}$*-manifold.*

We saw in Section 3.3 how κ may be thought (up to a constant multiplicative factor) as injectivity radius, now we see how $\sqrt{\kappa}$ may be thought (up to a constant multiplicative factor) as geodesic length. Unfortunately a clear physical interpretation of geodesic lengths is still lacking.

3.5. Antipodality

In a previous section we introduced the notion of antipodality in $\mathfrak{P}\mathcal{H}$ by making reference to the underlying linear structure. In this section we shall generalize this notion to arbitrary strongly nondegenerate Riemannian manifolds and we shall explain its physical meaning.

Definition 3.3. The *cut locus* C_x of any point $x \in M$ is the complement of the greatest open set M_x of M such that any point of M_x might be connected to x by means of one and only one minimal geodesic. We say that a pair of points $x, y \in M$ is *antipodal* if $x \in C_y$.

Obviously antipodality is a symmetric relation. From our study of the geodesic structure of $\mathfrak{P}\mathcal{H}$ we know that Definition 3.3 corresponds (in $\mathfrak{P}\mathcal{H}$) to our previous definition.

Now we shall explain the physical importance of antipodality. If $\hat{\varphi}, \hat{\chi}$ is any pair of points of $\mathfrak{P}\mathcal{H}$ our ability to distinguish them *with a single measurement* is given by the probability (see [8]):

$$p_{\max} = \tfrac{1}{2}\left(1 + \sqrt{1 - |(\varphi|\psi)|^2}\right). \tag{3.11}$$

We observe that $p_{\max}$ is minimum when $\varphi = \psi$ and maximum (actually one) when $\varphi \perp \psi$. So a pair of states is *totally distinguishable* if and only if they are antipodal.

4. The geodesic characterization of observables

In this section we will provide a geometric characterization of observables truly implementing the notion of linearity. The idea is to select the linear observables by means of their behavior on geodesics. Remark that geodesics reproduce on $\mathfrak{P}\mathcal{H}$ the lines of $\mathcal{H}$.

Definition 4.1. A smooth function $f : \mathfrak{P}\mathcal{H} \to \mathbb{R}$ is *geolinear* if and only if for any $\hat{\varphi} \in \mathfrak{P}\mathcal{H}$ and 2κ-normalized vector $v \in T_{\hat{\varphi}}\mathfrak{P}\mathcal{H}$ one has

$$f(c_{\hat{\varphi}v}(t)) = f(\hat{\varphi}) + (\sin t \cos t) d_{\hat{\varphi}} f(v) + (\sin^2 t)\mathrm{Hess}_{\hat{\varphi}} f(v). \tag{4.1}$$

Here $\mathrm{Hess}_{\hat{\varphi}} f(v) = \frac{1}{2}\mathrm{Hess}_{\hat{\varphi}} f(v, v)$ is the Hessian of f in $\hat{\varphi}$ evaluated on the pair (v, v). We refer the reader for an intrinsic definition of the Hessian to [29]. We point out that Definition 4.1 strongly depends on the specific nature of $\mathfrak{P}\mathcal{H}$:

- $\sin t$, $\cos t$ functions are specific to $\mathfrak{P}\mathcal{H}$-geodesics,
- the lack of third or higher degree terms is expected to be specific to $\mathfrak{P}\mathcal{H}$.

If $A : \mathcal{H} \to \mathcal{H}$ is any self-adjoint operator, the map

$$\langle A\rangle : \hat{\varphi} \in \mathfrak{P}\mathcal{H} \rightsquigarrow (\varphi|A\varphi) \in \mathbb{R}, \tag{4.2}$$

where $\varphi \in \hat{\varphi}$ is any normalized representative, is clearly geolinear. But a more strong result holds. Indeed we are now in a position to state the main result of this section:

Proposition 4.1. *A smooth map $f : \mathfrak{P}\mathcal{H} \to \mathbb{R}$ is geolinear if and only if there is a self-adjoint operator $A \in \mathcal{L}(\mathcal{H})$ such that $f = \langle A\rangle$.*

In the proof of this result we shall use the following trivial remark:

Remark 4.1. Let $A : \mathcal{H} \to \mathcal{H}$ be an $\mathbb{R}$-linear operator. $\langle A\rangle$ is well defined on $\mathfrak{P}\mathcal{H}$ if and only if A is $\mathbb{C}$-linear.

Proof of Proposition 4.1. Necessity is easy to prove, so we restrict ourselves to the verification of sufficiency. We divide the proof in two parts to make the proposition clearer.

Sufficiency (real Hilbert space). Let $f : \mathfrak{P}\mathcal{H} \to \mathbb{R}$ be a geolinear function, $\hat{\varphi}$ any element of $\mathfrak{P}\mathcal{H}$ and φ be a normalized representative of $\hat{\varphi}$. We shall use the symbol f to denote the local form of f with respect to the chart ξ_φ. The map $D^2\mathsf{f}(0) : \mathcal{H}_\varphi \times \mathcal{H}_\varphi \to \mathbb{R}$ is continuous, $\mathbb{R}$-linear and symmetric, so for a known property of Hilbert spaces there is a unique $\mathbb{R}$-linear continuous operator $B : \mathcal{H}_\varphi \to \mathcal{H}_\varphi$ such that

$$D^2\mathsf{f}(0)(\mathsf{v}, \mathsf{w}) = 2(\mathsf{v}|B\mathsf{w}). \tag{4.3}$$

Moreover, from Riesz' theorem we know that there is a unique vector $\zeta \in \mathcal{H}_\varphi$ such that

$$D\mathsf{f}(0)(\mathsf{v}) = 2(\zeta|\mathsf{v}). \tag{4.4}$$

Now we define $A : \mathcal{H} \to \mathcal{H}$ as the unique $\mathbb{R}$-linear operator such that

$$A\varphi := f(\hat{\varphi})\varphi + \zeta, \tag{4.5a}$$

$$A\eta := (\zeta|\eta)\varphi + [f(\hat{\varphi})1 + B]\eta, \tag{4.5b}$$

where η is any element of $\mathcal{H}_\varphi$. It is not difficult to verify that A is a continuous self-adjoint operator. To complete the proof one may now limit oneself to the easy verification of the three relations

$$\langle A\rangle(\hat{\varphi}) := f(\hat{\varphi}), \tag{4.6a}$$

$$d_{\hat{\varphi}}\langle A\rangle := d_{\hat{\varphi}} f, \tag{4.6b}$$

$$\mathrm{Hess}_{\hat{\varphi}}\langle A\rangle := \mathrm{Hess}_{\hat{\varphi}} f. \tag{4.6c}$$

Sufficiency (complex Hilbert space). Let $\mathcal{H}$ be a complex Hilbert space and $f : \mathfrak{P}\mathcal{H} \to \mathbb{R}$ a geolinear function. By restricting the scalar field and taking as scalar product $\Re(\ |\ )$ we obtain from $\mathcal{H}$ a real Hilbert space we shall denote with the symbol $\mathcal{H}_\Re$. The following diagram is obviously commutative (functions $\mathfrak{p}$, π, τ are canonical projections).

$$\begin{array}{ccc} \mathcal{H}\setminus\{0\} & \xrightarrow{\tau} & \mathfrak{P}\mathcal{H}_\Re \\ \mathfrak{p}\downarrow & & \downarrow f\circ\pi \\ \mathfrak{P}\mathcal{H} & \xrightarrow[f]{} & \mathbb{R} \end{array} \tag{4.7}$$

One easily sees that $f \circ \pi$ is geolinear on $\mathfrak{P}\mathcal{H}_\Re$. So we know from the first part of the proof that there is a unique $\mathbb{R}$-linear self-adjoint operator A such that $f \circ \pi = \langle A\rangle_\Re$, where

$$\langle A\rangle_\Re : \tau(\varphi) \in \mathfrak{P}\mathcal{H}_\Re \rightsquigarrow \Re(\varphi|A\varphi) \in \mathbb{R} \tag{4.8}$$

(φ is any normalized representative of $\hat{\varphi}$). Since f is well defined on $\mathfrak{P}\mathcal{H}$ and $f = \langle A\rangle_\Re$ (on $\mathfrak{P}\mathcal{H}_\Re$) also $\langle A\rangle_\Re$ is well defined on $\mathfrak{P}\mathcal{H}$. From the remark we deduce that A is $\mathbb{C}$-linear. Last from $\mathbb{C}$-linearity and real self-adjointness we deduce complex self-adjointness. □

5. The metric characterization of dynamics

We explained in Section 1 why we need to characterize in an independent way observables and dynamical vector fields, even if in ordinary QM these two concepts agree (Stone's theorem). Now we shall develop a complete characterization of dynamics starting from the Kählerian one given in [11] and improved in [12]:

Proposition 5.1. *A smooth map $f : \mathfrak{P}\mathcal{H} \to \mathbb{R}$ is Kählerian if and only there is a continuous linear operator $A : \mathcal{H} \to \mathcal{H}$ such that $f = \langle A \rangle$.*

Here *Kählerian* means that the flow of the Hamiltonian vector field v_f preserves all the Kählerian structure.

In this section we will show how, with a very simple remark, it is possible to improve the proposition deleting any reference to either the complex or the symplectic structure. Indeed we shall prove that dynamical vector fields are nothing more than the Killing ones, that is those whose flow is isometric.

We fix the following terminology:

- a *semi-unitary* operator on complex Hilbert space $\mathcal{H}$ is a linear unitary or antiunitary operator;
- an *antipodal* bijection on $\mathfrak{P}\mathcal{H}$ is a bijection $\hat{U} : \mathfrak{P}\mathcal{H} \to \mathfrak{P}\mathcal{H}$ such that both $\hat{U}$ and its inverse preserve antipodality;
- a homeomorphism $\hat{U} : \mathfrak{P}\mathcal{H} \to \mathfrak{P}\mathcal{H}$ *admits a conservative distance* α if for any pair of points $\hat{\varphi}, \hat{\psi}$ whose distance is α the distance between $\hat{U}(\hat{\varphi}), \hat{U}(\hat{\psi})$ is α (α is any positive real number);
- a diffeomorphism of $\mathfrak{P}\mathcal{H}$ is *isometric* if it preserves the Riemannian structure;
- a diffeomorphism of $\mathfrak{P}\mathcal{H}$ is *Kählerian* if it preserves the Riemannian and the symplectic structure.

In this section a fundamental role is played by Wigner's theorem in the strong formulation due to Uhlhorn (see [37, Theorem 5.1]) and Mielnik (see [28, Theorem 1]):

Theorem 5.1 (Wigner–Uhlhorn–Mielnik). *Let $\mathcal{H}$ be a Hilbert space and let the map $\hat{U}$: $\mathfrak{P}\mathcal{H} \to \mathfrak{P}\mathcal{H}$ be any bijection. If* $\dim \mathcal{H} \geq 3$, *the following assertions are equivalent:*

(a) *$\hat{U}$ is an antipodal diffeomorphism,*
(b) *$\hat{U}$ is a diffeomorphism admitting a conservative distance,*
(c) *$\hat{U}$ is an isometric diffeomorphism,*
(d) *$\hat{U}$ is a bijection preserving the transition probability,*
(e) *there is a semi-unitary operator U such that $\hat{U} = \mathfrak{P}U$.*

If $\dim \mathcal{H} < 3$ *then points* (c), (d) *and* (e) *are still equivalent but there are antipodality-preserving diffeomorphisms not implemented by any semi-unitary operator.*

The theorem of Wigner–Uhlhorn–Mielnik provides us with a complete characterization of semi-unitary operators in terms of the corresponding projective functions. A similar characterization for unitary operators is easy to find:

Corollary 5.1. *Let $\mathcal{H}$ be a Hilbert space and $\hat{U} : \mathfrak{P}\mathcal{H} \to \mathfrak{P}\mathcal{H}$ any bijection. If* $\dim \mathcal{H} \geq 3$ *the following assertions are equivalent:*
(a) *$\hat{U}$ is an antipodal biholomorphism,*
(b) *$\hat{U}$ is a biholomorphism admitting a conservative distance,*
(c) *$\hat{U}$ is a Kählerian diffeomorphism,*
(d) *$\hat{U}$ is a bijection preserving orientation and transition probabilities,*
(e) *there is a unitary operator U such that $\hat{U} = \mathfrak{P}U$.*
If $\dim \mathcal{H} < 3$ *then points* (c), (d) *and* (e) *are still equivalent.*

It is well known that the notion of orientation is not well defined on generic infinite-dimensional manifolds. So we specify that we refer to the notion of orientation on the pure states space of a C^*-algebra introduced by Alfsen, Hanche-Olsen and Shultz [36].

Let $\mathcal{P}$ be the group of Kählerian diffeomorhism of $\mathfrak{P}\mathcal{H}$. This group may be easily identified (and will be identified) with the projective group, that is the quotient of the unitary group $\mathfrak{U}$ with respect to its center. $\mathcal{P}$ endowed with the topology induced by the family of maps

$$p_{\varphi,\psi} : [U] \in \mathcal{P} \rightsquigarrow |(\varphi|U\psi)| \in \mathbb{R} \tag{5.1}$$

(with φ, ψ generic normalized vectors) is a topological group. The following is a standard result (see [5, Theorem 3.2] and [38]):

Lemma 5.1. *If $u : \mathbb{R} \to \mathcal{P}$ is a 1-parameter continuous group then there is a 1-parameter continuous group $U : \mathbb{R} \to \mathfrak{U}$ such that $u(t) = \mathfrak{P}U(t)$ for any t.*

A *Killing* vector field is a complete vector field whose flow preserves the Riemannian structure. We may now state the main result of this section:

Proposition 5.2. *A vector field v on $\mathfrak{P}\mathcal{H}$ is Killing if and only if there is an operator $H \in \mathcal{L}(\mathcal{H})_{sa}$ such that $v = v_{\langle H \rangle}$.*

Here we use the standard convention to denote with the symbol v_f the Hamiltonian vector field generated by the Hamiltonian f (in this case $f = \langle H \rangle$).

Proof of Proposition 5.2. Let v be a Killing vector field on $\mathfrak{P}\mathcal{H}$. Since its flow F is complete the maps F_t are globally defined isometric diffeomorphisms. From Wigner theorem we deduce the existence of a semi-unitary operator U_t such that $F_t := \mathfrak{P}U_t$. By virtue of the relation

$$F_t = F_{t/2} \circ F_{t/2} \tag{5.2}$$

this operator is actually unitary. Consequently U is a projective representation.

Since F is continuous the map

$$t \in \mathbb{R} \rightsquigarrow \mathfrak{p}(U_t\varphi) \in \mathfrak{P}\mathcal{H} \tag{5.3}$$

is continuous. From the definition of the topology of $\mathcal{P}$ we deduce the continuity of the map $t \in \mathbb{R} \rightsquigarrow \mathfrak{p}U_t \in \mathcal{P}$. Now we know that U is a strongly continuous unitary representation of $\mathbb{R}$ and then by virtue of Stone's theorem that there exists a (not necessarily bounded) self-adjoint operator $H \in \mathcal{L}(\mathcal{H})_{sa}$ such that

$$F(t, \hat{\varphi}) = \mathfrak{p} \exp\left(-\frac{\mathrm{i}}{\kappa} t H\right). \tag{5.4}$$

Since the left-hand member of this equation is smooth, this is also true for the right-hand member. By exploiting the continuity of the time derivative (in 0) of the right-hand member of Eq. (5.4) it is not difficult to show that H is a bounded operator. Given this remark by differentiating Eq. (5.4) we obtain the desired relation $v = v_{\langle H \rangle}$. □

Remark 5.1. We point out that Eq. (5.4) tells us that the flow of the Killing vector field $v_{\langle H \rangle}$ is the (projectivization of) the solution of the Schrödinger equation with Hamiltonian H if and only if $\kappa = \hbar$. So *while any positive value of κ is acceptable from an abstract point of view, only $\hbar$ gives us a theory well corresponding to ordinary QM.*

6. The superposition principle

6.1. An informal description

Superposing plays a relevant role among techniques of state generation because it is the only one always preserved by quantum dynamics and observations. The superposition principle is nothing more than a nontriviality assumption: any pair of (different) pure states ρ, σ has at least a nontrivial superposition ν (that is $\nu \neq \rho, \sigma$). As Wick et al. have shown in [41] the QSP has a restricted validity: not any pair of pure states may be superposed. A maximal set of superposable states is called superselection sector.

6.2. A formal description

In the abstract framework of quantum logic axiomatics any physical system is described by means of a set of properties L (logic) and a set of pure states P. Usually L is an orthoposet set and P is a set of probability measures on L, moreover there is a duality allowing us to speak about values of an element $p \in P$ on an element $a \in L$. In such an abstract setting two main notions of superposition have been introduced. Gudder [20] says: a state $\nu \in P$ is a *superposition* of two states $\rho, \sigma \in P$ if and only if for any $a \in L$

$$\rho(a) = \sigma(a) = 0 \quad \Longrightarrow \quad \nu(a) = 0. \tag{6.1}$$

Guz' alternative definition [21] may be obtained by simply substituting Eq. (6.1) with

$$\rho(a) = \sigma(a) = 1 \quad \Longrightarrow \quad \nu(a) = 1. \tag{6.2}$$

Physically Guz' definition tell us that *ν is a superposition of ρ, σ only when any property true in ρ and σ is also true in ν.* A similar interpretation holds for Gudder's definition. It is relevant from a geometrical point of view to remark that Gudder's definition may be equivalently formulated, in our setting of ordinary QM in a given superselection sector, using positive operators rather than elements of the logic (orthogonal projectors). Similarly Guz' definition may be equivalently formulated taking effects rather than elements of the logic. Having a notion of superposition we may state the QSP. Among the various formulations which have been proposed (see [16,20,23,33,21]) we choose Guz [21] and Pulmannova [33] for its physical and geometrical clearness: *the QSP holds for a pair* (L, P) *if and only if any pair of (distinct) pure states has at least a nontrivial superposition.*

6.3. The geometric description

It is a well-known fact that for any pair of (distinct) points $\hat{\varphi}$, $\hat{\chi}$ of $\mathfrak{P}\mathcal{H}$ the set of all superpositions of these two states is nothing more than the projective of the two-dimensional subspace of $\mathcal{H}$ generated by any pair of representatives of $\hat{\varphi}$, $\hat{\chi}$. From a metric point of view this is the smallest geodesically closed set generated by $\hat{\varphi}$ and $\hat{\chi}$. This suggests to look at geodesics as a way to implement superpositions in a more general geometric setting.

In a very abstract way we may think of a manifold M with a distinguished set of geodesics as being the (pure) state space of some physical system in a generalized quantum theory. A typical situation is the following. The manifold M is endowed with a connection whose (projection on M of) integral curves we call geodesics. There is some way to select a class of geodesics we call minimal. For instance if F is a Riemannian connection, we select geodesics parametrized by arc length. More generally, on a Finslerian manifold we may take as minimal geodesics those with normalized velocity.

In this abstract setting we state:

Definition 6.1. For any $x, y \in M$: any point of the smallest geodesically closed set containing x and y (resp. a geodesic connecting x and y) is a (ordinary) *superposition* (resp. *strong* superposition) of x and y.

So the geometrical formulation of QM gives us not only a way to describe superpositions but also a very natural two-level classification. Obviously enough we are left with the problem to identify a set of physical criterions allowing us to distinguish among these two types of superposition.

Since geodesics implement superpositions it is natural to think of the lack of geodesics as a way to implement superselection rules:

Definition 6.2. A *superselection sector* (in M) is a subset C of M such that for any $x \in C$ and $y \notin C$ there is no geodesic connecting x and y. We say that M *satisfies the QSP* if it has only one superselection sector.

Remark 6.1. In projective QM antipodality is strictly linked with geodesic degeneration, indeed a pair of points is antipodal if and only if there is a ($U(1)$-parametrized) family of geodesics connecting these two points. In geometric QM (with superselection rules) there is also a second type of antipodality, indeed a pair of points geodesically disconnected is always antipodal. Since antipodality is a mathematical implementation of total (physical) distinguishability we have: *there is a distinguished way to superpose states (strong superpositions) only when the states are not totally distinguishable.*

6.4. Observables

In projective QM observables are nothing more than geolinear functions. So it is natural to think of geolinearity as a way to describe observables in our abstract setting of QM on an arbitrary manifold M. Unfortunately, geolinearity strongly depends on $\sin t$, $\cos t$ functions, which are a peculiarity of the geodesic structure of $\mathfrak{P}\mathcal{H}$. We remark that if $f : \mathfrak{P}\mathcal{H} \to \mathbb{R}$ is geolinear, then *the knowledge of f in a neighborhood of any (fixed) point $\hat{\varphi} \in \mathfrak{P}\mathcal{H}$ allows us to know that the value of f is any point of $\mathfrak{P}\mathcal{H}$.* This property is fundamental but too weak to uniquely characterize observables. Moreover, this property is not sufficient to establish a linkage between superpositions and observables of the type described by Eqs. (6.1) and (6.2). A second property of any geolinear function f is the following: *the value of f in any point $\hat{\chi}$ of any geodesic connecting two (distinct) points $\hat{\varphi}$, $\hat{\psi}$ is a linear combinations of $f(\hat{\varphi})$, $f(\hat{\psi})$, $d_{\hat{\varphi}} f(v)$ (for some vector $v \in T_{\hat{\varphi}}\mathfrak{P}\mathcal{H}$) or equivalently $d_{\hat{\psi}} f(w)$ (for some vector $w \in T_{\hat{\psi}}\mathfrak{P}\mathcal{H}$).* We remark that if we take these two properties as a definition of observable in the abstract setting of (generalized) QM on an arbitrary manifold M then positive (valued) observables always satisfy condition (6.1). So our definition is enough rich to ensure the usual interplay between positive observables and superpositions but we have no proof that on $\mathfrak{P}\mathcal{H}$ this definition uniquely characterizes geolinear functions.

6.5. Dynamics

Let $\hat{U} : \mathfrak{P}\mathcal{H} \to \mathfrak{P}\mathcal{H}$ be any superposition preserving bijection, i.e. such that for any pair of points $\hat{\varphi}$, $\hat{\psi}$ there are points φ', ψ' such that

$$\hat{U}\widehat{\varphi\psi} = \widehat{\varphi'\psi'}. \tag{6.3}$$

As remarked by Gisin [18] the fundamental theorem of projective geometry tell us that there exists a semi-linear operator $U : \mathcal{H} \to \mathcal{H}$ such that $\hat{U} = \mathfrak{P}U$. Moreover, Gisin has shown that if $\hat{U}$ is continuous, then U is either linear or antilinear. So *a superposition preserving homeomorphism is the projectivization of a continuous linear (or antilinear) automorphism of $\mathcal{H}$.*

Now let $\hat{U}$ be a strong-superposition preserving bijection, that is a map sending any point of a geodesic connecting $\hat{\varphi}$, $\hat{\psi}$ in a point of a (same) geodesic connecting $\hat{U}\hat{\varphi}$, $\hat{U}\hat{\psi}$ (we point out that in this section we think of geodesics as set of points, that is we identify geodesics coinciding up to reparametrization). Since $\hat{U}$ is bijective the 1-parameter family of geodesics connecting a pair of (generic) antipodal points $\hat{\varphi}$, $\hat{\psi}$ is mapped onto the 1-parameter family of geodesics connecting the points $\hat{U}\hat{\varphi}$, $\hat{U}\hat{\psi}$. Only antipodal points may be connected by different geodesics, so $\hat{U}$ preserves antipodality. Hence *$\hat{U}$ is the projectivization of a unitary (or antiunitary) operator.*

Now we recall that geometrically an angle is a pair of vectors tangent in a same point and that on $\mathfrak{P}\mathcal{H}$ any two vectors v, w tangent in the same point $\hat{\varphi}$ define two angles

$$\angle(v_{\hat{\varphi}}, w_{\hat{\varphi}}) := \arccos g_{\hat{\varphi}}(v_{\hat{\varphi}}, w_{\hat{\varphi}}),$$
$$\measuredangle(v_{\hat{\varphi}}, w_{\hat{\varphi}}) := \arccos \omega_{\hat{\varphi}}(v_{\hat{\varphi}}, w_{\hat{\varphi}}).$$

It is clear from our previous analysis that any strong-superposition preserving bijection always preserves angles $\angle$. Moreover, one sees easily that such a bijection preserves angles $\measuredangle$ if and only if it is the projectivization of a unitary operator. Given two geodesics c and e connecting a pair of antipodal points $\hat{\varphi}$, $\hat{\chi}$, we may take the vectors tangent to these geodesics in any of these two antipodal points as a way to define an angle between geodesics. So we shall say angle between a pair of geodesics (connecting a pair of antipodal points) meaning angle between these vectors. If one takes angle $\angle$, the definition does not depend on the chosen antipodal point, while the $\measuredangle$ has opposite values in $\hat{\varphi}$ and $\hat{\chi}$. Hence we may state: *a strong-superposition preserving bijection preserves $\measuredangle$-angles between geodesics if and only if it is the projectivization of a unitary operator.*

7. The uncertainty principle

A widespread statement is that quantum mechanics sets severe limits to our ability to simultaneously measure physical quantities. This statement is usually supported by means of this phenomenological principle: *there is no simultaneous measurement of the quantities q, p in which the product of the errors is less than $\hbar/2$.* As a formalization of this principle one usually takes the inequality:

$$\Delta q \Delta p \geq \frac{\hbar}{2}. \tag{7.1}$$

But it has correctly been stressed that this inequality is not equivalent to the above phenomenological principle. So it is fundamental to point out that speaking about uncertainty relations we are referring to Eq. (7.1) (with the analogous relations for the other phase space coordinates).

7.1. Uncertainty relations among observables

Given the Hilbertian formulation of QM the uncertainty relations among observables become a simple mathematical theorem. Indeed Robertson (see [34]) has proved that:

Heisenberg inequality. For any $A, B \in \mathcal{L}(\mathcal{H})_{sa}$ and $\varphi \in \mathcal{H}$

$$\Delta_\varphi A \Delta_\varphi B \geq \tfrac{1}{2}|(\varphi|[A, B]\varphi)|, \tag{7.2}$$

where

$$\Delta_\varphi A := \|A\varphi - (\varphi|A\varphi)\varphi\| \tag{7.3}$$

is the dispersion of the observable A in the state φ.

The translation of this inequality in the language of differential geometry on $\mathfrak{P}\mathcal{H}$ has been performed in [12,13]. We recall the main points. For any pair of smooth functions $f, h : \mathfrak{P}\mathcal{H} \to \mathbb{R}$ let be

$$\{f, h\}(\hat{\varphi}) := \omega_{\hat{\varphi}}(v_f(\hat{\varphi}), v_h(\hat{\varphi})), \tag{7.4a}$$

$$((f, h))(\hat{\varphi}) := g_{\hat{\varphi}}(v_f(\hat{\varphi}), v_h(\hat{\varphi})), \tag{7.4b}$$

where v_f denotes the Hamiltonian vector field generated by f. Then one proves easily that:

Proposition 7.1. *For any $A, B \in \mathcal{L}(\mathcal{H})_{sa}$*
– $\{\langle A\rangle, \langle B\rangle\} = \langle \frac{1}{i\kappa}[A, B]\rangle$,
– $((\langle A\rangle\langle B\rangle)) = \frac{2}{\kappa}\langle A \circ B\rangle - \frac{2}{\kappa}\langle A\rangle\langle B\rangle$,
where $A \circ B$ is the Jordan product of A and B.

So the Poissonian product $\{\_,\_\}$ implements at a geometric level the Lie algebra structure of the observables and the Riemannian product $((\_,\_))$ (pointwisely deformed) implements the Jordan algebra structure. One easily sees that

$$\Delta^2_{\varphi} A = \frac{\kappa}{2}((\langle A\rangle_{\hat{\varphi}}, \langle A\rangle_{\hat{\varphi}})) = \frac{\kappa}{2}\|v_{\langle A\rangle}(\hat{\varphi})\|^2_g. \tag{7.5}$$

Thus we may rewrite Eq. (7.2) as

$$|\{\langle A\rangle(\hat{\varphi}), \langle B\rangle(\hat{\varphi})\}| \le \|v_{\langle A\rangle}(\hat{\varphi})\|_g \|v_{\langle B\rangle}(\hat{\varphi})\|_g, \tag{7.6}$$

or equivalently

$$|\omega_{\hat{\varphi}}(v_{\langle A\rangle}(\hat{\varphi}), v_{\langle B\rangle}(\hat{\varphi}))| \le \|v_{\langle A\rangle}(\hat{\varphi})\|_g \|v_{\langle B\rangle}(\hat{\varphi})\|_g. \tag{7.7}$$

Since any element of $T_{\hat{\varphi}}\mathfrak{P}\mathcal{H}$ may be written in the form $v_{\langle A\rangle}(\hat{\varphi})$, for some operator A, we may equivalently rewrite (7.7) in the form

$$\|\omega_{\hat{\varphi}}\| \le 1. \tag{7.8}$$

So up to renormalization Heisenberg inequality is nothing more than the pointwise continuity of the simplectic form with respect to the the topology induced on the tangent space by the Finslerian (Riemannian) structure. In a more general context we could say that Heisenberg inequality corresponds to asking for continuity of the Poissonian structure with respect to the Finslerian topology in the detailed meaning coded in Eq. (7.6).

Up to now we were speaking about ordinary (projective) QM. Now we are ready to delinearize and to introduce the uncertainty principle in a general geometric setting.

Definition 7.1. Let M be a manifold endowed with a Poissonian structure $\{\_,\_\}$ and a Finslerian structure $\|\_\|$. We say that $(M, \{\_,\_\}, \|\_\|)$ *satisfies the Uncertainty Principle* if the Poissonian structure is pointwisely continuous with respect to the Finslerian topology: there is some $a \in \mathbb{R}^+$ such that for any pair of smooth functions $f, h : M \to \mathbb{R}$

$$|\{f, h\}(x)| \le a\|v_f(x)\|\|v_h(x)\|, \tag{7.9}$$

fo any $x \in M$.

Indeed if (7.9) is satisfied and we let

$$\Delta_x^2 f := \frac{\kappa}{2} a \| v_f(x) \| \tag{7.10}$$

then

$$\frac{\kappa}{2} |\{f, h\}(x)| \leq \Delta_x f \Delta_x h. \tag{7.11}$$

On any manifold satisfying the uncertainty principle, and then on $\mathfrak{P}\mathcal{H}$ too, the uncertainty inequality (7.11) holds for any pair of smooth functions and not only for the geolinear ones. So Eq. (7.11) has a deeper meaning than Eq. (7.6). We point out that (7.11) shows that *Heisenberg inequality does not depend in any way on the linearity of observables.*

Remark 7.1. Let M be a manifold endowed with a strongly nondegenerate symplectic tensor ω and a strongly nondegenerate Riemannian tensor g, such that (M, ω, g) satisfies the uncertainty principle. For any point x the bilinear form ω_x is antisymmetric, nondegenerate and continuous with respect to the Riemannian topology on tangent space $T_x M$. A known theorem (see [1, Theorem 3.1.19]) tells us that there exist a complex structure J_x and a real inner product $\Re(\_|\_)$ on $T_x M$ such that

$$(v|w) := \Re(v|w) - J_x \omega_x(v, w) \tag{7.12}$$

is a Hermitian nondegenerate inner product. So in this restricted context it is correct to state that to satisfy the uncertainty principle we are compelled to introduce (pointwisely) a complex structure. But we remark that, as far as we are concerned with ordinary formulation of the uncertainty principle, we are not compelled to ask either for smoothness of J or for a Kählerian linkage between ω and g.

7.2. The dynamical formulation of the uncertainty principle

Heisenberg inequality is clearly a constraint on the values one may obtain by measuring a pair of observables. We shall show in this section that we may also see it as a constraint on the relative portrait of dynamical flows. Let A, B be continuous self-adjoint operators and $\hat{\varphi}$ any element of $\mathfrak{P}\mathcal{H}$. If c, e are smooth curves on $\mathfrak{P}\mathcal{H}$ such that $c(t) = e(t) = \hat{\varphi}$, $\dot{c}(t) = v_{\langle A \rangle}[c(t)]$ and $\dot{e}(t) = v_{\langle B \rangle}[e(t)]$, then we may rewrite Eq. (7.7) as

$$|\omega_{\hat{\varphi}}(\dot{c}(t), \dot{e}(t))| \leq \|\dot{c}(t)\|_g \|\dot{e}(t)\|_g, \tag{7.13}$$

which is equivalent to

$$|\cos \measuredangle(\dot{c}(t), \dot{e}(t))| \leq \|\dot{c}(t)\|_g \|\dot{e}(t)\|_g. \tag{7.14}$$

Since $\|\dot{c}(t)\|_g$ (resp. $\|\dot{e}(t)\|_g$) is the *velocity* of c (resp. of e) in t one has the following geometric interpretation of (7.14): *the product of the velocities of c and e in their intersection points is always greater than the cosine of the intersection angle of the two curves.* In a more suggestive way we could equivalently say that in order that c and e were both slow in $\hat{\varphi}$ it is necessary that their intersection angle is great. So *in quantum mechanics the dynamical* *portrait of a Hamiltonian vector field constrains the form of the flow of any Hamiltonian vector field generated by an observable not commuting with the original one.*

7.3. Aharonov–Anandan time-energy uncertainty inequality

Let $\mathrm{h} : \mathfrak{P}\mathcal{H} \to \mathbb{R}$ be any smooth function and c any integral curve of the Hamiltonian vector field v_{h}. If $\hat{\varphi} := c(t_0)$, $\hat{\chi} := c(t_1)$, with $t_1 > t_0$, then

$$\int_{t_0}^{t_1} \mathrm{d}t\, g_{c(t)}(\dot{c}(t), \dot{c}(t))^{1/2} \geq d(\hat{\varphi}, \hat{\chi}). \tag{7.15}$$

Since $\dot{c}(t) = v_{\mathrm{h}}[c(t)]$ and

$$\Delta^2_{c(t)}\mathrm{h} = \frac{\kappa}{2}\|v_{\mathrm{h}}[c(t)]\|^2_g, \tag{7.16}$$

we have

$$\int_{t_0}^{t_1} \mathrm{d}t\, \Delta_{c(t)}\mathrm{h} \geq \sqrt{\frac{\kappa}{2}} d(\hat{\varphi}, \hat{\chi}). \tag{7.17}$$

Let

$$\langle \Delta_{c(t)}\mathrm{h} \rangle := \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} \mathrm{d}t\, \Delta_{c(t)}\mathrm{h}, \tag{7.18a}$$

$$\Delta t := t_1 - t_0. \tag{7.18b}$$

If

$$\langle \Delta t \rangle := \frac{\sqrt{2\kappa}(\pi/2)}{d(\hat{\varphi}, \hat{\chi})}\Delta t \tag{7.19}$$

then (7.17) becomes

$$\langle \Delta_{c(t)}\mathrm{h} \rangle \langle \Delta t \rangle \geq \frac{\pi}{2}\kappa. \tag{7.20}$$

In ordinary QM $\kappa = \hbar$ so the right member of (7.20) becomes $h/4$. Hence (7.20) is the time-energy inequality derived by Anandan in [3] as a generalization of a previous result [2]. Mathematically (7.20) is clear but the interpretation of $\langle \Delta t \rangle$ as a *time incertitude* seems to us a little disputable. So we adopt a different viewpoint. Equivalently Eq. (7.17) may be rewritten as

$$\frac{\langle \Delta_{c(t)}\mathrm{h} \rangle}{\langle \mathrm{v} \rangle} \geq \frac{\kappa}{2}, \tag{7.21}$$

where

$$\langle \mathrm{v} \rangle := \frac{d(\hat{\varphi}, \hat{\chi})}{\Delta t}. \tag{7.22}$$

We give to (7.21) the following interpretation: *a physical system whose dynamics is described by the (nonlinear) Hamiltonian* h *may be energetically well collimated (on the average) only if its (mean) projective velocity is small.*

It is not difficult to see that (7.21) is an equality only when c is a geodesic. This allows us to understand better the differences among strong and ordinary superpositions: *dynamics along strong superpositions has the highest degree of (mean) energetic collimation for a given (mean) projective velocity.*

8. Conclusions

As far as we are concerned with the mathematical formalism of quantum mechanics at an abstract level, we are not compelled to restrict ourselves to projective Hilbert spaces. We only need a manifold endowed with a Finslerian structure, a connection and a Poissonian structure. The geodesical structure describes superpositions and the continuity of the Poissonian structure gives HUP.

Of course, such a general schema takes into account just of some fundamental requirements for a quantum theory and there are many open questions. Therefore, at this stage of development of our program we do not think that the above simple schema, as it is, could be a possible framework for the geometrical description of nonlinear quantum mechanics. While it is clear that the HUP requires continuity of the Poissonian product with respect to the Finslerian topology, no much is known about the relations between this connection and the Finslerian (resp. Poissonian) structure. That is we do not know which physical principles forbid us to fix in an arbitrary way the relations between these structures. In [12,13], given a manifold endowed with a strongly nondegenerate Riemannian tensor and a strongly nondegenerate symplectic tensor, a mix of physical and mathematical motivations was singled out to reconstruct the projective structure of quantum mechanics. Giving up to these mathematical constraints a large plethora of nonlinear extensions of QM becomes possible. These extensions are too many and too weak to be used in a phenomenological analysis. Further inspection and the introduction of some other fundamental principles is necessary to select 'a' nonlinear extension of QM. In particular, QSP and HUP are stricly connected with spectral theory of observables. So we have to select, on the basis of phenomenological requirements, some nonlinear observables admitting a suitable spectral theory. In a forthcoming publication we shall explain how further restriction may be obtained asking for a spectral theory and a probabilistic interpretation of nonlinear observables.

Note added in proof

After submitting this article an interesting paper by Ashtekar and Schilling [4] has appeared. By virtue of the relevance of this work we think as necessary to give a brief confrontation. Ashtekar and Schilling take as an abstract model for a nonlinear quantum system a Kähler manifold M and as observables those functions $f : M \to \mathbb{R}$ whose flow is Killing.

Given these hypotheses they obtain a theorem (Theorem III.1) stating that under reasonable conditions on the observable set the manifold necessarily has constant holomorphic sectional curvature. So under standard topological hypothesis, at least in the finite dimensional case, the Kähler manifold necessarily is the projective of a Hilbert space. This work is much in the spirit of [12] (see Proposition 4.6) where, starting from a most general class of manifolds and using a different set of hypothesis, a similar result was obtained. The spirit of the present work is strongly different. We look for the most abstract model consistently admitting a formulation of two of the most fundamental aspects of QM: the superposition principle and the Heisenberg uncertainty principle. So doing we obtain that there is no need to restrict ourselves to the class of Kähler manifolds. Furthermore, we take as separate notions those of observable and dynamical vector field obtaining different characterizations. We remark that stating that Killing vector fields are nothing more than ordinary quantum dynamical vector fields is much more than saying that observables are those functions whose Hamiltonian vector field is Killing. Indeed we are also stating the nonobvious fact that any Killing vector field is also Hamiltonian.

References

[1] R. Abraham, J.E. Marsden, Foundations of Mechanics, Addison-Wesley, Reading, MA, 1978.

[2] Y. Aharonov, J. Anandan, Geometry of quantum evolution, Phys. Rev. Lett., 65 (1990) 1697–1700.

[3] J. Anandan, A geometric approach to quantum mechanics, Foundations Phys. 21 (1991) 1265–1284.

[4] A. Ashtekar, T.A. Schilling, Geometrical formulation of quantum mechanics, gr-qc, 9706069, 1997.

[5] V. Bargmann, On unitary ray representations of continuous groups, Ann. Math. 59 (1954) 1–46.

[6] A.L. Besse, Manifolds all of whose Geodesics are Closed, Springer, New York, 1978.

[7] N. Bourbaki, Algèbre, Chapitre III: Algèbre Multilinéaire, Hermann, Paris, 1958.

[8] P. Busch, Is the quantum state (an) observable?, in: Experimental Metaphysics – Quantum Mechanical Studies in Honor of Abner Shimony, D. Reidel, Dordrecht, 1996.

[9] V. Cantoni, Superpositions of physical states: a metric viewpoint, Helvetica Phys. Acta 58 (1985) 956–968.

[10] S.S. Chern, Finsler geometry is just Riemannian geometry without the quadratic restriction, Notices AMS, September 1996.

[11] R. Cirelli, P. Lanzavecchia, Hamiltonian vector fields in quantum mechanics, Il Nuovo Cimento B 79 (1984) 271–283.

[12] R. Cirelli, A. Manià, L. Pizzocchero, Quantum mechanics as an infinite-dimensional Hamiltonian system with uncertainty structure: Part I, J. Math. Phys. 31 (1990) 2891–2897.

[13] R. Cirelli, A. Manià, L. Pizzocchero, Quantum mechanics as an infinite-dimensional Hamiltonian system with uncertainty structure: Part II, J. Math. Phys. 31 (1990) 2898–2903.

[14] L. Dabrowski, A superposition principle for mixed states?, Il Nuovo Cimento. B 106 (1991) 963–968.

[15] L. de Broglie, Non-Linear Wave Mechanics. A causal interpretation, Elsevier, Amsterdam, 1960.

[16] G. Emch, C. Piron, Symmetry in quantum theory, J. Math. Phys. 4 (1963) 469–473.

[17] M. Gatti, Sull'estensione non lineare dei Principi fondamentali della Meccanica Quantistica, Ph.D. Thesis, Università Statale di Milano, 1997.

[18] N. Gisin, Irreversible quantum dynamics and the Hilbert space structure of quantum kinematics, J. Math. Phys. 24 (1983) 1779–1782.

[19] N. Gisin, Weinberg's non-linear quantum mechanics and supraluminal communications, Phys. Lett. A 143 (1990) 1–2.

[20] S. Gudder, A superposition principle in physics, J. Math. Phys. 11 (1970) 1037–1040.

[21] W. Guz, Projection postulate and superposition principle in non-lattice quantum logics, Ann. Inst. Henri Poincaré 34 (1981) 373–389.

[22] A. Heslot, Quantum mechanics as a classical theory, Phys. Rev. D 31 (1985) 1341–1348.

[23] J.M. Jauch, Foundations of Quantum Mechanics, Addison-Wesley, Reading, MA, 1968.
[24] T.W.B. Kibble, S. Randjbar-Daemi, Non-linear coupling of quantum theory and classical gravity, J. Phys. A 13 (1980) 141–148.
[25] S. Lang, Differential and Riemannian Manifolds, Springer, New York, 1995.
[26] J.E. Marsden, T.S. Ratiu, Introduction to Mechanics and Symmetry, Springer, New York, 1994.
[27] B. Mielnik, Generalized quantum mechanics, Comm. Math. Phys. 37 (1974) 221–256.
[28] B. Mielnik, Phenomenon of mobility in non-linear theories, Commun. Math. Phys. 101 (1985) 323–339.
[29] R. Palais, C. Terng, Critical Point Theory and Submanifold Geometry, Springer, New York, 1988.
[30] R.S. Palais, Lusternik–Schnirelman theory on Banach manifolds, Topology 5 (1966) 115–132.
[31] A. Peres, Nonlinear variants of Schrödinger's equation violate the second law of thermodynamics, Phys. Rev. Lett. 63 (1989) 1114.
[32] A.D. Popova, Nonlinear quantum mechanics with nonclassical gravitational self-interaction, Internat. J. Modern Phys. 4 (1989) 3229–3267.
[33] S. Pulmannova, A superposition principle in quantum logics, Commun. Math. Phys. 49 (1976) 47–51.
[34] H.P. Robertson, The uncertainty principle, Phys. Rev. 34 (1929) 163–164.
[35] D.J. Rowe, A. Ryman, G. Rosensteel, Many-body quantum mechanics as a symplectic dynamical system, Phys. Rev. A 22 (1980) 2362–2373.
[36] F.W. Shultz, Pure states as a dual object for C^*-algebras, Commun. Math. Phys. 82 (1982) 497–509.
[37] U. Uhlhorn, Representation of simmetry transformations in quantum mechanics, Arkiv för Fysik 23 (1962) 307–340.
[38] V.S. Varadarajan, Geometry of Quantum Theory, vol. II, D, van Nostrand, Toronto, 1970.
[39] S. Weinberg, Testing quantum mechanics, Ann. Phys. 194 (1989).
[40] S. Weinberg, Weinberg replies, Phys. Rev. Lett. 63 (1989) 1115, Reply to Peres article: Nonlinear variants of Schrödinger's equation violate the second law of thermodynamics.
[41] G.C. Wick, A.S. Wightman, E.P. Wigner, The intrinsic parity of elementary particles, Phys. Rev. 88 (1952) 101–105